

# The Bivariate Lack-of-Memory Distributions

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**Abstract.** We first review the univariate and bivariate lack-of-memory properties (LMPs). The univariate LMP is a remarkable characterization of the exponential distribution, while the bivariate LMP is shared by the famous Marshall and Olkin's, Block and Basu's as well as Freund's bivariate exponential distributions. We treat all the bivariate lack-of-memory (BLM) distributions in a unified approach and develop some new general properties of the BLM distributions, including joint moment generating function, product moments and dependence structure. Necessary and sufficient conditions for the survival functions of BLM distributions to be totally positive of order two are given. Some previous results for specific BLM distributions are improved. In particular, we show that both the Marshall–Olkin survival copula and survival function are totally positive of all orders, regardless of parameters. Besides, we point out that Slepian's inequality also holds true for the BLM distributions.

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## 1. Introduction

The classical univariate lack-of-memory property (LMP) is a remarkable characterization of the exponential distribution which plays a prominent role in reliability theory, queuing theory and other applied fields (Feller 1965, Fortet 1977, Galambos and Kotz 1978). The recent bivariate LMP is, however, shared by the famous Marshall and Olkin's, Block and Basu's as well as Freund's bivariate exponential distributions, among many others; see, e.g., Chapter 10 of Balakrishnan and Lai (2009), Chapter 47 of Kotz, Balakrishnan and Johnson (2000) and Kulkarni (2006). These bivariate distributions have been well investigated individually in the literature. Our main purpose in this paper is, however, to develop in a unified approach some new general properties of the bivariate lack-of-memory (BLM) distributions which share the same bivariate LMP.

In Section 2, we briefly review the univariate LMP and its ramifications. In Section 3, we review the bivariate LMP and summarize the important known properties of the BLM distributions. Then we derive in Section 4 some new general properties of the BLM distributions, including joint moment generating function, product moments and stochastic inequalities. The dependence structures of the BLM distributions are investigated in Section 5. We find necessary and sufficient conditions for the survival functions (and the densities if they exist) of BLM distributions to be totally positive of order two. Some previous results for specific BLM distributions are improved. In particular, we show that both the Marshall–Olkin survival copula and survival function are totally positive of all orders, regardless of parameters. In Section 6, we study the stochastic comparisons in the family of all BLM distributions and point out that Slepian's lemma/inequality on bivariate normal distributions also holds true for the BLM distributions.

## 2. Univariate Lack-of-Memory Property

We briefly review the well-known univariate lack-of-memory property. Let  $X$  be a non-negative random variable with distribution function  $F$ . Then  $F$  satisfies (multiplicative) Cauchy's functional equation

$$\overline{F}(x+y) = \overline{F}(x)\overline{F}(y), \quad x \geq 0, y \geq 0, \quad (1)$$

where  $\overline{F}(x) = 1 - F(x) = \Pr(X > x)$ , if and only if  $F(0) = 1$  ( $X$  degenerates at 0) or  $F(x) =$

$1 - \exp(-\lambda x)$ ,  $x \geq 0$ , for some constant  $\lambda > 0$ , denoted by  $X \sim \text{Exp}(\lambda)$  ( $X$  has an exponential distribution with positive parameter  $\lambda$ ). If  $X$  is the lifetime of a system with positive survival function  $\overline{F}$ , then Eq (1) is equivalent to

$$\Pr[X > x + y | X > y] = \Pr[X > x], \quad x \geq 0, y \geq 0. \quad (2)$$

This means that the conditional probability of a system surviving to time  $x+y$  given surviving to time  $y$  is equal to the unconditional probability of the system surviving to time  $x$ . Namely, the failure performance of the system does not depend on the past, given its present condition. In such a case (2), we say that the distribution  $F$  lacks memory at each point  $y$ . So Eq (1) is called the lack-of-memory property (LMP) or memoryless property of  $F$ .

The LMP (1) is a remarkable characterization of the exponential distribution. On the other hand, suppose the random variable  $X$  takes only nonnegative integer values, then  $X \sim F$  satisfies the functional equation (instead of (1)):  $\overline{F}(x+y) = \overline{F}(x)\overline{F}(y)$ ,  $x, y = 0, 1, 2, \dots$ , if and only if (i)  $X$  degenerates at one point 0 or 1, or (ii)  $\Pr(X = n) = (1-p)p^{n-1}$ ,  $n = 1, 2, \dots$ , where  $p = \overline{F}(1) \in (0, 1)$  (note that in this paper we define  $F(x) = \Pr(X \leq x)$  rather than  $\Pr(X < x)$ ). Namely, the geometric distribution is the counterpart of the exponential distribution in the discrete case. Once we have a characterization result about the exponential distribution, there exists usually a corresponding characteristic property of the geometric distribution.

For simplicity, we consider only *positive* random variable  $X \sim F$  from now on. Then, the LMP (1) holds true iff  $X \sim \text{Exp}(\lambda)$  for some  $\lambda > 0$ .

We next review some ramifications of the LMP.

(a) *Almost-LMP:  $F$  lacks memory at one point  $c$ .* We have that for some fixed  $c > 0$ ,

$$\overline{F}(c+x) = \overline{F}(c)\overline{F}(x), \quad x \geq 0, \quad (3)$$

if and only if  $F(x) = 1 - \alpha^{[x/c]} + \alpha^{[x/c]}F(x - [x/c]c)$  for  $x > c$ , where  $\alpha = \overline{F}(c) < 1$ ,  $[x]$  is the integer part of  $x$  and  $F$  has no other requirements in  $(0, c]$ . To see this, we note that Eq (3) holds true iff  $\overline{F}(nc+x) = \overline{F}(nc)\overline{F}(x)$ ,  $x \geq 0$ ,  $n = 1, 2, \dots$  ( $F$  lacks memory at each point  $nc$ ,  $n = 1, 2, \dots$ ) iff  $\overline{F}(nc+x) = \alpha^n \overline{F}(x)$ ,  $x \geq 0$ ,  $n = 1, 2, \dots$ . Therefore, (3) is called an almost-LMP of  $F$  in the literature. In this case,  $\overline{F}$  looks like a periodic function but with

weight  $\alpha^n$  decreasing to 0 as  $n \rightarrow \infty$ .

(b) *F lacks memory at two points with incommensurable values.* Let  $c_1$  and  $c_2$  be two positive constants with  $c_1/c_2$  irrational. Then  $X \sim F$  satisfies the functional equation

$$\overline{F}(c+x) = \overline{F}(c)\overline{F}(x), \quad c = c_1, c_2, \quad x \geq 0,$$

iff  $X \sim \text{Exp}(\lambda)$  for some  $\lambda > 0$ .

(c) *F lacks memory at a random point.* Let  $X$  and  $Y$  be independent positive random variables. Under suitable conditions on  $X$  and  $Y$ , we have that  $\Pr[X > Y + z] = \Pr[X > Y] \Pr[X > z]$ ,  $z \geq 0$ , iff  $X \sim \text{Exp}(\lambda)$  for some  $\lambda > 0$ .

(d) *More general result.* Let  $X, Y$  and  $Z$  be independent positive random variables. Under suitable conditions on  $X, Y$  and  $Z$ , we have that  $\Pr[X > Y + Z] = \Pr[X > Y] \Pr[X > Z]$  iff  $X \sim \text{Exp}(\lambda)$  for some  $\lambda > 0$ .

(e) *Density form and its variants.* Let  $X > 0$  have a density  $f$  and  $f(0^+) = \lim_{x \rightarrow 0^+} f(x)$ . Then  $f(x+y)f(0^+) = f(x)f(y)$ ,  $x, y > 0$ , iff  $X \sim \text{Exp}(\lambda)$  for some  $\lambda > 0$ .

For details of the above variants of LMP, see Marsaglia and Tubilla (1975), Chukova and Dimitrov (1992), Lin (1994), Shimizu (1979), Ramachandran (1977), Huang (1979), Obretenov (1970), Krishnaji (1971), and Roy and Roy (2013). For more variants of LMP, see Bairamov (2000), Prakasa Rao (2004), Stepniak (2009), Dimitrov, Rykov and Krougly (2004), Shimizu (1978), Dimitrov, Chukova and Green (1997), and Sandhya and Rajasekharan (2012).

### 3. Bivariate Lack-of-Memory Property

We now consider the bivariate lack-of-memory property (BLMP). Let the positive random variables  $X$  and  $Y$  have joint distribution  $H$  with marginals  $F$  and  $G$ . Namely,  $(X, Y) \sim H$ ,  $X \sim F$ ,  $Y \sim G$ . Moreover, denote the survival function of  $H$  by

$$\overline{H}(x, y) \equiv \Pr(X > x, Y > y) = 1 - F(x) - G(y) + H(x, y), \quad x, y \geq 0.$$

An intuitive extension of the LMP (2) to the bivariate case is the strict bivariate LMP:

$$\Pr(X > x + s, Y > y + t | X > s, Y > t) = \Pr(X > x, Y > y), \quad x, y, s, t \geq 0$$

( $H$  lacks memory at each pair  $(s, t)$ ), which is equivalent to

$$\overline{H}(x + s, y + t) = \overline{H}(x, y)\overline{H}(s, t), \quad \forall x, y, s, t \geq 0, \tag{4}$$

if the survival function  $\overline{H}$  is positive. In a two-component system, this means as before that the conditional probability of two components surviving to times  $(x+s, y+t)$  given surviving to times  $(s, t)$  is equal to the unconditional probability of these two components surviving to times  $(x, y)$ . But Eq (4) has only one solution, namely, the independent bivariate exponential distribution,

$$\overline{H}(x, y) = \exp[-(\lambda x + \delta y)], \quad x, y \geq 0,$$

for some constants  $\lambda, \delta > 0$ ; in other words,  $X$  and  $Y$  are independent random variables and  $X \sim \text{Exp}(\lambda)$ ,  $Y \sim \text{Exp}(\delta)$  for some positive parameters  $\lambda, \delta$ .

In their pioneering paper, Marshall and Olkin (1967) considered instead the weaker bivariate LMP (with  $s = t$ )

$$\Pr(X > x+t, Y > y+t \mid X > t, Y > t) = \Pr(X > x, Y > y), \quad x, y, t \geq 0$$

( $H$  lacks memory at each *equal* pair  $(t, t)$ ), and solved the functional equation

$$\overline{H}(x+t, y+t) = \overline{H}(x, y)\overline{H}(t, t), \quad \forall x, y, t \geq 0. \quad (5)$$

It turns out that for given  $(X, Y) \sim H$  with marginals  $F, G$  on  $(0, \infty)$ , the BLMP (5) is equivalent to

$$\overline{H}(x, y) = \begin{cases} e^{-\theta y} \overline{F}(x-y), & x \geq y \geq 0 \\ e^{-\theta x} \overline{G}(y-x), & y \geq x \geq 0 \end{cases} \quad (6)$$

for some constant  $\theta > 0$ .

Like the univariate case, there are many variants of BLMP; see, e.g., Ghurye (1987), Roy and Mukherjee (1989, 1998), Roy (2002), Roy and Roy (2014), Dimitrov, Chukova and Khalil (1995), Nair and Nair (1988), Obretenov (1985), John (2011), Pinto and Kolev (2015), and Marshall and Olkin (2015). But we are not going to pursue all the variants here. For simplicity, we will focus only on the bivariate lack-of-memory (BLM) distribution  $H$  defined in (6), although some of the obtained results can be extended to higher dimensional settings.

For convenience, denote by  $BLM(F, G, \theta)$  the BLM distribution  $H$  with marginals  $F, G$ , parameter  $\theta > 0$  and survival function  $\overline{H}$  in (6). Theorem 1 below summarizes some important known properties of the BLM distributions; for more details, see Marshall and Olkin (1967), Block and Basu (1974), Block (1977), and Ghurye and Marshall (1984).

**Theorem 1.** Let  $(X, Y) \sim H = BLM(F, G, \theta)$ . Then the following statements are true.

- (i) The marginals  $F, G$  have densities  $f, g$ , respectively. Moreover, the right-hand derivatives  $f(x) = \lim_{\varepsilon \rightarrow 0+} [F(x+\varepsilon) - F(x)]/\varepsilon$  and  $g(x) = \lim_{\varepsilon \rightarrow 0+} [G(x+\varepsilon) - G(x)]/\varepsilon$  exist for all  $x \geq 0$ , which are right-continuous and are of bounded variation on  $[0, \infty)$ .
- (ii)  $\Pr(X - Y > t) = \overline{F}(t) - f(t)/\theta$  and  $\Pr(Y - X > t) = \overline{G}(t) - g(t)/\theta$  for all  $t \geq 0$ .
- (iii) Both  $e^{\theta x} f(x)$  and  $e^{\theta x} g(x)$  are increasing (nondecreasing) in  $x \geq 0$ .
- (iv)  $F(x) + G(x) \geq 1 - \exp(-\theta x)$ ,  $x \geq 0$ .
- (v)  $X \wedge Y \sim \text{Exp}(\theta)$  and is independent of  $X - Y$ , where  $X \wedge Y = \min\{X, Y\}$ .
- (vi)  $f(0) \vee g(0) \leq \theta \leq f(0) + g(0)$ , where  $f(0) \vee g(0) = \max\{f(0), g(0)\}$ .
- (vii)  $f'(x) + \theta f(x) \geq 0$ ,  $g'(x) + \theta g(x) \geq 0$ ,  $x \geq 0$ , if  $f$  and  $g$  are differentiable.

**Remark 1.** Some of the above necessary conditions (i)–(vii) also play as sufficient conditions for  $(X, Y)$  to have a BLM distribution. For example, in addition to the above conditions (vi) and (vii), assume that the marginal densities are absolutely continuous, then the  $\overline{H}$  in (6) is a *bona fide* survival function. This is a slight modification of Theorem 5.1 of Marshall and Olkin (1967) who required (vi')  $[f(0) + g(0)]/2 \leq \theta \leq f(0) + g(0)$  instead of (vi) above. Note that conditions (vi) and (vi') are different unless  $f(0) = g(0)$ , and that (vi) is a consequence of (iii) and (iv) (see Corollary 2(i) below and Ghurye and Marshall 1984, p.792). Two more observations are that at least one of  $f(0)$  and  $g(0)$  is positive, and that the survival function  $\overline{H}$  in (6) is purely singular (i.e.,  $X = Y$  almost surely) iff  $\theta = [f(0) + g(0)]/2$  iff  $f(0) = g(0) = \theta$  (because  $f(0) \neq g(0)$  implies  $\theta > [f(0) + g(0)]/2$  by (vi)). On the other hand, the condition (v) together with continuous marginals  $F, G$  also implies that  $(X, Y)$  has a BLM distribution (Block 1977, p.810). It is interesting to recall that for *independent* nondegenerate random variables  $X$  and  $Y$ , the above independence of  $X \wedge Y$  and  $X - Y$  is a characterization of the exponential/geometric distributions under suitable conditions (see Ferguson 1964, 1965, Crawford 1966, and Rao and Shanbhag 1994). Namely, in general, the BLM distributions share the same independence property of  $X \wedge Y$  and  $X - Y$  with independent exponential/geometric random variables. Recall also that the  $H = BLM(F, G, \theta)$  is absolutely continuous (i.e.,  $X \neq Y$  almost surely) iff its marginal densities together satisfy  $f(0) + g(0) = \theta$  (see Ghurye and Marshall 1984, p.792).

**Remark 2.** Kulkarni (2006) proposed an interesting and useful approach to construct some BLM distributions by starting with marginal failure rate functions. First, choose two real-valued functions  $r_1, r_2$  and a constant  $\theta$  satisfying the following (modified) conditions:

- (a) The functions  $r_i, i = 1, 2$ , are *absolutely continuous* on  $[0, \infty)$  and  $\theta > 0$ .
- (b)  $0 \leq r_i(x) \leq \theta, x \geq 0, i = 1, 2$ .
- (c)  $\int_0^\infty r_i(x)dx = \infty, i = 1, 2$ .
- (d)  $r_i(x)(\theta - r_i(x)) + r_i'(x) \geq 0, x \geq 0, i = 1, 2$ .
- (e)  $r_1(0) + r_2(0) \geq \theta$ .

Then set  $\overline{F}(x) = \exp(-\int_0^x r_1(t)dt), x \geq 0$ , and  $\overline{G}(y) = \exp(-\int_0^y r_2(t)dt), y \geq 0$ . In this way, the  $H$  defined through (6) is a *bona fide* BLM distribution because the above conditions (a)–(e) together imply that conditions (vi) and (vii) in Theorem 1 hold true (see Remark 1).

We now briefly review three important BLM distributions in the literature. For more details, see, e.g., Chapter 10 of Balakrishnan and Lai (2009).

**Example 1.** Marshall and Olkin's (1967) bivariate exponential distribution (BVE)

If both marginals  $F$  and  $G$  are exponential, then the  $BLM(F, G, \theta)$  defined in (6) reduces to the Marshall–Olkin BVE with survival function of the form:

$$\overline{H}(x, y) = \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max\{x, y\}] \quad (7)$$

$$\equiv \frac{\lambda_1 + \lambda_2}{\lambda} \overline{H}_a(x, y) + \frac{\lambda_{12}}{\lambda} \overline{H}_s(x, y), \quad x, y \geq 0, \quad (8)$$

where  $\lambda_1, \lambda_2, \lambda_{12}$  are positive constants,  $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$ , and  $H_a, H_s$  (written explicitly below) are absolutely continuous and singular bivariate distributions, respectively. In practice, the Marshall–Olkin BVE arises from the following shock model. Let us consider a two-component system and suppose that there are three independent sources of fatal shocks present in the environment:

- (i) A shock from source 1 destroys component 1; it occurs at random time  $X_1 \sim \text{Exp}(\lambda_1)$ .
- (ii) A shock from source 2 destroys component 2; it occurs at random time  $X_2 \sim \text{Exp}(\lambda_2)$ .
- (iii) Finally, a shock from source 3 destroys both components; it occurs at random time  $X_3 \sim \text{Exp}(\lambda_{12})$ .

Then the lifetimes of two components are  $(X, Y) = (X_1 \wedge X_3, X_2 \wedge X_3)$  and have a joint survival function  $\overline{H}$  defined in (7). The singular part in (8) is identified by the conditional

probability:  $\overline{H}_s(x, y) = \Pr(X > x, Y > y | X_3 \leq X_1 \wedge X_2) = \exp[-\lambda \max\{x, y\}]$ , while the absolutely continuous part  $\overline{H}_a$  is calculated from  $\overline{H}$  and  $\overline{H}_s$  via (8) (see the next example).

**Example 2.** Block and Basu's (1974) bivariate exponential distribution

The Block–Basu BVE is actually the absolute continuous part  $H_a$  of Marshall–Olkin BVE in (8) and has a joint density of the form

$$h(x, y) = \begin{cases} \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \exp[-(\lambda_1 + \lambda_{12})x - \lambda_2 y], & x \geq y > 0 \\ \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \exp[-\lambda_1 x - (\lambda_2 + \lambda_{12})y], & y > x > 0, \end{cases} \quad (9)$$

where  $\lambda_1, \lambda_2, \lambda_{12} > 0$ , and  $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$ . Its survival function is equal to

$$\begin{aligned} \overline{H}(x, y) &= \overline{H}_a(x, y) \\ &= \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max\{x, y\}] - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} \exp[-\lambda \max\{x, y\}], \quad x, y \geq 0. \end{aligned}$$

Note that in this case, the marginals are not exponential but rather *negative mixtures* of two exponentials. Specifically,  $\overline{F}(x) = \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-(\lambda_1 + \lambda_{12})x] - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} \exp(-\lambda x)$ ,  $x \geq 0$ , and  $\overline{G}(y) = \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-(\lambda_2 + \lambda_{12})y] - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} \exp(-\lambda y)$ ,  $y \geq 0$ .

**Example 3.** Freund's (1961) bivariate exponential distribution

The Freund BVE has a joint density of the form

$$h(x, y) = \begin{cases} \alpha' \beta \exp[-(\alpha + \beta - \alpha')y - \alpha' x], & x \geq y > 0 \\ \alpha \beta' \exp[-(\alpha + \beta - \beta')x - \beta' y], & y > x > 0, \end{cases} \quad (10)$$

where  $\alpha, \alpha', \beta, \beta' > 0$ . If  $\alpha + \beta > \alpha' \vee \beta'$ , its survival function is equal to

$$\overline{H}(x, y) = \begin{cases} \frac{\beta}{\alpha + \beta - \alpha'} \exp[-(\alpha + \beta - \alpha')y - \alpha' x] + \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} \exp[-(\alpha + \beta)x], & x \geq y \geq 0 \\ \frac{\alpha}{\alpha + \beta - \beta'} \exp[-(\alpha + \beta - \beta')x - \beta' y] + \frac{\beta - \beta'}{\alpha + \beta - \beta'} \exp[-(\alpha + \beta)y], & y \geq x \geq 0. \end{cases}$$

To derive the Freund BVE, let us consider the following two-component system. Suppose the only dependence between two components arises from failure of either component changing the parameter of the life distribution of the other component in the following way:

- (i) Initially, components 1 and 2 have exponential life distributions with parameters  $\alpha, \beta$ , respectively.
- (ii) When component 1 fails, the  $\beta$  for component 2 becomes  $\beta'$ .
- (iii) When component 2 fails, the  $\alpha$  for component 1 becomes  $\alpha'$ .

Then the lifetimes  $(X, Y)$  of two components have a joint density function  $h$  given in (10).



It worths noting that by choosing  $\alpha = \frac{\lambda_1 \lambda}{\lambda_1 + \lambda_2}$ ,  $\beta = \frac{\lambda_2 \lambda}{\lambda_1 + \lambda_2}$ ,  $\alpha' = \lambda_1 + \lambda_{12}$  and  $\beta' = \lambda_2 + \lambda_{12}$ , Freund's BVE (10) reduces to Block and Basu's BVE (9).

#### 4. New General Properties of Bivariate LM Distributions

Let  $(X, Y) \sim H = BLM(F, G, \theta)$  with marginals  $F$  and  $G$  on  $(0, \infty)$ , parameter  $\theta > 0$  and survival function (6). Denote the Laplace transform of  $X$  by  $L_X$ , and that of  $(X, Y)$  by  $\mathcal{L}$ . Then we have

**Theorem 2.** The Laplace transform of  $(X, Y) \sim H = BLM(F, G, \theta)$  is

$$\mathcal{L}(s, t) \equiv E(e^{-sX-tY}) = \frac{1}{\theta + s + t} [(\theta + s)L_X(s) + (\theta + t)L_Y(t)] - \frac{\theta}{\theta + s + t}, \quad s, t > 0.$$

To prove Theorem 2, we need the following lemma due to Lin, Lai and Govindaraju (2016).

**Lemma 1.** Let  $(X, Y) \sim H$  defined on  $R_+^2 = [0, \infty) \times [0, \infty)$ . Then the Laplace transform of  $(X, Y)$  is equal to  $\mathcal{L}(s, t) = st \int_0^\infty \int_0^\infty \overline{H}(x, y) e^{-sx-ty} dx dy - 1 + L_X(s) + L_Y(t)$ ,  $s, t \geq 0$ .

**Proof of Theorem 2.** We have to calculate the double integral

$$\int_0^\infty \int_0^\infty \overline{H}(x, y) e^{-sx-ty} dx dy = \int \int_{x \geq y} + \int \int_{y \geq x} \equiv A_1 + A_2,$$

where, by changing variables and by integration by parts,

$$\begin{aligned} A_1 &= \int_0^\infty e^{-(\theta+t)y} \int_y^\infty e^{-sx} \overline{F}(x-y) dx dy = \int_0^\infty e^{-(\theta+s+t)y} \int_0^\infty e^{-sz} \overline{F}(z) dz dy \\ &= \frac{1}{\theta + s + t} \left[ -\frac{1}{s} \int_0^\infty \overline{F}(z) d e^{-sz} \right] = \frac{1}{\theta + s + t} \left[ \frac{1}{s} (1 - L_X(s)) \right], \end{aligned}$$

and similarly,

$$A_2 = \frac{1}{\theta + s + t} \left[ \frac{1}{t} (1 - L_Y(t)) \right].$$

Lemma 1 together with the above  $A_1$  and  $A_2$  completes the proof.

Denote the moment generating function (mgf) of  $X$  by  $M_X$ , and that of  $(X, Y)$  by  $\mathcal{M}$ . Then we have the following general result.

**Theorem 3.** Let  $(X, Y) \sim H = BLM(F, G, \theta)$  and let  $r, s$  be real numbers such that  $s + t < \theta$ . Then the mgf of  $(X, Y)$  is

$$\mathcal{M}(s, t) \equiv E(e^{sX+tY}) = \frac{1}{\theta - s - t} [(\theta - s)M_X(s) + (\theta - t)M_Y(t)] - \frac{\theta}{\theta - s - t},$$

provided the expectations (mgfs) exist.

To prove Theorem 3, we need instead the following lemma due to Lin, Dou, Kuriki and Huang (2014).

**Lemma 2.** Let  $(X, Y) \sim H$  defined on  $R_+^2$ . Let  $\alpha$  and  $\beta$  be two increasing and left-continuous functions on  $R_+$ . Then the expectation of the product  $\alpha(X)\beta(Y)$  is equal to

$$E[\alpha(X)\beta(Y)] = \int_0^\infty \int_0^\infty \overline{H}(x, y) d\alpha(x) d\beta(y) - \alpha(0)\beta(0) + \alpha(0)E[\beta(Y)] + \beta(0)E[\alpha(X)],$$

provided the expectations exist.

**Proof of Theorem 3.** Case (i):  $s, t \geq 0$ . Let  $\alpha(x) = e^{sx}$  and  $\beta(y) = e^{ty}$  in Lemma 2, then

$$\mathcal{M}(s, t) = st \int_0^\infty \int_0^\infty \overline{H}(x, y) e^{sx+ty} dx dy - 1 + M_X(s) + M_Y(t).$$

We have to calculate the double integral

$$\int_0^\infty \int_0^\infty \overline{H}(x, y) e^{sx+ty} dx dy = \iint_{x \geq y} + \iint_{y \geq x} \equiv B_1 + B_2,$$

where, by changing variables and by integration by parts,

$$\begin{aligned} B_1 &= \int_0^\infty e^{-(\theta-t)y} \int_y^\infty e^{sx} \overline{F}(x-y) dx dy = \int_0^\infty e^{-(\theta-s-t)y} \int_0^\infty e^{sz} \overline{F}(z) dz dy \\ &= \frac{1}{\theta-s-t} \left[ \frac{1}{s} \int_0^\infty \overline{F}(z) d e^{sz} \right] = \frac{1}{\theta-s-t} \left[ \frac{1}{s} (-1 + M_X(s)) \right], \end{aligned}$$

and similarly,

$$B_2 = \frac{1}{\theta-s-t} \left[ \frac{1}{t} (-1 + M_Y(t)) \right].$$

Lemma 2 together with the above  $B_1$  and  $B_2$  completes the proof of Case (i).

Case (ii):  $s \geq 0, t < 0$ . To apply Lemma 2, set  $\alpha(x) = e^{sx}$  and  $\beta(y) = 1 - e^{ty}$ . Then both  $\alpha$  and  $\beta$  are increasing functions on  $R_+$  and  $E[\alpha(X)\beta(Y)] = M_X(s) - \mathcal{M}(s, t)$ . Therefore,

$$\mathcal{M}(s, t) = M_X(s) - E[\alpha(X)\beta(Y)] = M_X(s) - 1 + M_Y(t) + st \int_0^\infty \int_0^\infty \overline{H}(x, y) e^{sx+ty} dx dy.$$

As before, we carry out the above double integral and complete the proof of Case (ii).

Case (iii):  $s < 0, t \geq 0$ . Set  $\alpha(x) = 1 - e^{sx}$  and  $\beta(y) = e^{ty}$  in Lemma 2. The remaining proof is similar to that of Case (ii) and is omitted.

Case (iv):  $s, t < 0$ . This case was treated in Theorem 1. The proof is completed.

Next, we consider the product moments of BLM distributions.

**Theorem 4.** For positive integers  $i$  and  $j$ , the product moment  $E[X^i Y^j]$  of  $(X, Y) \sim H = BLM(F, G, \theta)$  is of the form

$$E[X^i Y^j] = i j \sum_{k=0}^{i-1} \frac{1}{i-k} \binom{i-1}{k} \frac{\Gamma(j+k)}{\theta^{j+k}} E(X^{i-k}) + i j \sum_{k=0}^{j-1} \frac{1}{j-k} \binom{j-1}{k} \frac{\Gamma(i+k)}{\theta^{i+k}} E(Y^{j-k}),$$

provided the expectations exist.

The first product moment has a neat representation in terms of marginal means and the parameter  $\theta$ , from which we can calculate Pearson's correlation of BLM distributions.

**Corollary 1.**  $E(XY) = \frac{1}{\theta}[E(X) + E(Y)]$  provided the expectations exist.

To prove Theorem 4 above, we will apply the following lemma due to Lin, Dou, Kuriki and Huang (2014).

**Lemma 3.** Let  $(X, Y) \sim H$  defined on  $R_+^2$ , and let the expectations  $E[X^r Y^s]$ ,  $E[X^r]$  and  $E[Y^s]$  be finite for some positive real numbers  $r$  and  $s$ . Then the product moment

$$E[X^r Y^s] = rs \int_0^\infty \int_0^\infty \overline{H}(x, y) x^{r-1} y^{s-1} dx dy.$$

**Proof of Theorem 4.** We have to calculate the double integral

$$\int_0^\infty \int_0^\infty \overline{H}(x, y) x^{i-1} y^{j-1} dx dy = \iint_{x \geq y} + \iint_{y \geq x} \equiv C_1 + C_2,$$

where, by changing variables and by integration by parts,

$$\begin{aligned} C_1 &= \int_0^\infty e^{-\theta y} y^{j-1} \int_y^\infty x^{i-1} \overline{F}(x-y) dx dy = \int_0^\infty e^{-\theta y} y^{j-1} \int_0^\infty (y+z)^{i-1} \overline{F}(z) dz dy \\ &= \sum_{k=0}^{i-1} \binom{i-1}{k} \int_0^\infty y^{j-1+k} e^{-\theta y} \int_0^\infty z^{i-1-k} \overline{F}(z) dz dy \\ &= \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{\Gamma(j+k)}{\theta^{j+k}} \left[ \frac{1}{i-k} \int_0^\infty \overline{F}(z) dz^{i-k} \right] = \sum_{k=0}^{i-1} \frac{1}{i-k} \binom{i-1}{k} \frac{\Gamma(j+k)}{\theta^{j+k}} E(X^{i-k}), \end{aligned}$$

and similarly,

$$C_2 = \sum_{k=0}^{j-1} \frac{1}{j-k} \binom{j-1}{k} \frac{\Gamma(i+k)}{\theta^{i+k}} E(X^{j-k}).$$

Finally, Lemma 3 together with the above  $C_1$  and  $C_2$  completes the proof.

For moment generating functions of *individual* BLM distributions, see Chapter 47 of Kotz, Balakrishnan and Johnson (2000), while for product moments of such distributions, see Nadarajah (2006).

For the next and later results, we need some notations in reliability theory. For random variables  $X \sim F$  and  $Y \sim G$ , we say that  $X$  is smaller than  $Y$  in the usual stochastic order (denoted by  $X \leq_{st} Y$ ) if  $\overline{F}(x) \leq \overline{G}(x)$  for all  $x$ , that  $X$  is smaller than  $Y$  in the hazard rate order (denoted by  $X \leq_{hr} Y$ ) if  $\overline{G}(x)/\overline{F}(x)$  is increasing in  $x$ , and that  $X$  is smaller than  $Y$  in the reversed hazard rate order (denoted by  $X \leq_{rh} Y$ ) if  $G(x)/F(x)$  is increasing in  $x$ . Suppose  $F$  and  $G$  have densities  $f$  and  $g$ , respectively. Then we say that  $X$  is smaller than  $Y$  in the likelihood ratio order (denoted by  $X \leq_{lr} Y$ ) if  $g(x)/f(x)$  is increasing in  $x$ . For more definitions of the related stochastic orders, see, e.g., Müller and Stoyan (2002), Shaked and Shanthikumar (2007) as well as Lai and Xi (2006).

On the other hand, for a distribution  $F$  itself we say that

- (a)  $F$  is increasing failure rate (IFR) if  $-\log \overline{F}(x)$  is convex in  $x \geq 0$ , or, equivalently,  $r(x) = f(x)/\overline{F}(x)$  is increasing in  $x \geq 0$  when the density  $f$  of  $F$  exists,
- (b)  $F$  is decreasing failure rate (DFR) if  $-\log \overline{F}(x)$  is concave in  $x \geq 0$ ,
- (c)  $F$  is increasing failure rate in average (IFRA) if  $-(1/x) \log \overline{F}(x)$  is increasing in  $x > 0$ , or, equivalently,  $\overline{F}^\alpha(x) \leq \overline{F}(\alpha x)$  for all  $\alpha \in (0, 1)$  and  $x \geq 0$ , and
- (d)  $F$  is decreasing failure rate in average (DFRA) if  $\overline{F}^\alpha(x) \geq \overline{F}(\alpha x)$  for all  $\alpha \in (0, 1)$  and  $x \geq 0$ . (See Barlow and Proschan 1981, Chapters 3 and 4.)

The bivariate IFRA and bivariate DFRA distributions  $H$  are defined similarly:

$H$  is bivariate IFRA if  $\overline{H}^\alpha(x, y) \leq \overline{H}(\alpha x, \alpha y)$  for all  $\alpha \in (0, 1)$  and  $x, y \geq 0$ ;  $H$  is bivariate DFRA if the above inequality for  $\overline{H}$  is reversed (see Block and Savits 1976, 1980).

Using reliability language, we have the following useful results.

**Theorem 5.** Let  $(X, Y) \sim H = BLM(F, G, \theta)$  and  $Z \sim Exp(\theta)$ . Then

- (i)  $Z \leq_{lr} X$  and  $Z \leq_{lr} Y$ ;
- (ii)  $Z \leq_{st} X$  and  $Z \leq_{st} Y$ ;  $Z \leq_{hr} X$  and  $Z \leq_{hr} Y$ ;  $Z \leq_{rh} X$  and  $Z \leq_{rh} Y$ ;
- (iii)  $(X, Y)$  has a bivariate IFRA distribution iff both marginals  $F$  and  $G$  are IFRA;
- (iv)  $(X, Y)$  has a bivariate DFRA distribution iff both marginals  $F$  and  $G$  are DFRA.

**Proof.** Part (i) follows immediately from Theorem 1(iii) (see Ghurye and Marshall 1984), while part (ii) follows from the fact that the likelihood ratio order is stronger than the usual stochastic order, hazard rate order and reversed hazard rate order (Müller and Stoyan

2002, pp. 12–13). Part (iii) holds true by verifying that  $\overline{H}(\alpha x, \alpha y) \geq \overline{H}^\alpha(x, y) \forall \alpha \in (0, 1), x, y \geq 0$ , iff (a)  $\overline{F}(\alpha x) \geq \overline{F}^\alpha(x) \forall \alpha \in (0, 1), x \geq 0$ , and (b)  $\overline{G}(\alpha y) \geq \overline{G}^\alpha(y) \forall \alpha \in (0, 1), y \geq 0$ . The proof of part (iv) is similar.

Applying the above stochastic inequalities, we can simplify the proof of some previous known results. For example, we have

**Corollary 2.** Let  $(X, Y) \sim H = BLM(F, G, \theta)$ . Then the following statements are true.

- (i) Both hazard rates of marginals  $F, G$  are bounded by  $\theta$  and hence  $\theta \geq f(0) \vee g(0)$ .
- (ii) Both the functions  $F(-\frac{1}{\theta} \log(1 - t))$  and  $G(-\frac{1}{\theta} \log(1 - t))$  are convex in  $t \in [0, 1)$ , and hence  $f'(x) + \theta f(x) \geq 0, g'(x) + \theta g(x) \geq 0, x \geq 0$ , if  $f$  and  $g$  are differentiable.
- (iii) Let  $S_F, S_G$  be the supports of marginals  $F, G$  with densities  $f, g$ , respectively. Then  $S_F = [a_F, \infty), S_G = [a_G, \infty)$  for some nonnegative constants  $a_F, a_G$  with  $a_F a_G = 0$  and  $f, g$  are positive on  $(a_F, \infty), (a_G, \infty)$ , respectively.
- (iv) If  $H$  is not absolutely continuous, then  $f(0) > 0, g(0) > 0$  and hence  $a_F = a_G = 0$ .

**Proof.** Part (i) follows from the facts that  $Z \leq_{hr} X$  and  $Z \leq_{hr} Y$ , where  $Z \sim Exp(\theta)$ , while part (ii) is due to the probability-probability plot characterization for  $Z \leq_{lr} X$  and  $Z \leq_{lr} Y$  (see Theorem 1.4.3 of Müller and Stoyan 2002). Part (iii) follows from that  $Z \leq_{lr} X, Z \leq_{lr} Y$  and Theorem 1(iv) because the latter implies that at least one of the left extremities  $a_F$  and  $a_G$  of marginal distributions should be zero. Finally, to prove part (iv), we note that  $\Pr(X - Y > 0) = 1 - f(0)/\theta$  and  $\Pr(Y - X > 0) = 1 - g(0)/\theta$  by Theorem 1(ii) (see Ghurye and Marshall 1984, p. 789). So if  $H$  is not absolutely continuous,  $\Pr(X = Y) > 0$  and hence  $f(0) = \theta \Pr(X \leq Y) > 0$  and  $g(0) = \theta \Pr(Y \leq X) > 0$ . The proof is complete.

## 5. Dependence Structures of Bivariate LM Distributions

Recall that a bivariate distribution  $H$  with marginals  $F$  and  $G$  is positively quadrant dependent (PQD) if

$$H(x, y) \geq F(x)G(y) \quad \forall x, y \geq 0, \quad \text{or, equivalently,} \quad \overline{H}(x, y) \geq \overline{F}(x)\overline{G}(y) \quad \forall x, y \geq 0.$$

A stronger (positive dependence) property than the PQD is the total positivity defined below. For a real-valued function  $K$  on the rectangle  $(a, b) \times (c, d)$ , we say that  $K(x, y)$  is totally positive of order  $r$  ( $TP_r, r \geq 2$ ) in  $x$  and  $y$  if for each fixed  $s \in \{2, 3, \dots, r\}$  and for all  $a < x_1 < x_2 < \dots < x_s < b$  and  $c < y_1 < y_2 < \dots < y_s < d$ , the determinant of the

$s \times s$  matrix  $(K(x_i, y_j))$  is nonnegative. The function  $K$  is said to be  $TP_\infty$  if it is  $TP_r$  for any order  $r \geq 2$  (Karlin 1968).

We now characterize the  $TP_2$  property of the survival functions of BLM distributions.

**Theorem 6.** Let  $(X, Y) \sim H = BLM(F, G, \theta)$ . Then the survival function  $\bar{H}$  is  $TP_2$  iff the marginal distributions  $F$  and  $G$  are IFR and satisfy  $\bar{F}(x)\bar{G}(x) \leq \exp(-\theta x)$ ,  $x \geq 0$ .

**Proof.** We define the cross-product ratio of  $\bar{H}$ :

$$r \equiv r(x_1, x_2; y_1, y_2) = \frac{\bar{H}(x_1, y_1)\bar{H}(x_2, y_2)}{\bar{H}(x_1, y_2)\bar{H}(x_2, y_1)}, \quad 0 < x_1 < x_2, \quad 0 < y_1 < y_2.$$

Then, by definition,  $\bar{H}$  is  $TP_2$  iff  $r(x_1, x_2; y_1, y_2) \geq 1$  for all  $0 < x_1 < x_2$ ,  $0 < y_1 < y_2$ .

(Necessity) Suppose that  $\bar{H}$  is  $TP_2$ . Then for all  $0 < x_1 = y_1 < x_2 = y_2$ , we have

$$r = r(x_1, x_2; x_1, x_2) = \frac{\bar{H}(x_1, x_1)\bar{H}(x_2, x_2)}{\bar{H}(x_1, x_2)\bar{H}(x_2, x_1)} = \frac{\exp(-\theta(x_2 - x_1))}{\bar{F}(x_2 - x_1)\bar{G}(x_2 - x_1)} \geq 1.$$

This implies that  $\bar{F}(x)\bar{G}(x) \leq \exp(-\theta x)$ ,  $x \geq 0$ . Next, we prove that the marginal distribution  $G$  is IFR, that is,  $g(y)/\bar{G}(y)$  is increasing in  $y \geq 0$ , or, equivalently,

- (i)  $\frac{\bar{G}(y+t)}{\bar{G}(t)}$  is decreasing in  $t \in (0, \infty)$  for each  $y \geq 0$  (Barlow and Proschan 1981, p. 54), or
- (ii)  $\frac{\bar{G}(t)}{\bar{G}(y+t)}$  is increasing in  $t \in (0, \infty)$  for each  $y \geq 0$ , or
- (iii)  $\frac{\bar{G}(y-x_2)}{\bar{G}(y-x_1)}$  is increasing in  $y > x_2$  for any fixed  $0 < x_1 < x_2$ , or
- (iv) the ratio  $r_G^* \equiv \frac{\bar{G}(y_1-x_1)\bar{G}(y_2-x_2)}{\bar{G}(y_2-x_1)\bar{G}(y_1-x_2)} \geq 1$  for all  $0 < x_1 < x_2 < y_1 < y_2$ ,

which is true because in this case  $r_G^* = r(x_1, x_2; y_1, y_2) \geq 1$  by (6) and the assumption.

Similarly, we can prove that  $F$  is IFR because the ratio

$$r_F^* \equiv \frac{\bar{F}(x_1 - y_1)\bar{F}(x_2 - y_2)}{\bar{F}(x_2 - y_1)\bar{F}(x_1 - y_2)} \geq 1 \quad \text{for all } 0 < y_1 < y_2 < x_1 < x_2.$$

(Sufficiency) Suppose that the marginal distributions  $F$  and  $G$  are IFR and satisfy  $\bar{F}(x)\bar{G}(x) \leq \exp(-\theta x)$ ,  $x \geq 0$ . Then we want to prove that  $\bar{H}$  is  $TP_2$ , that is, for all  $0 < x_1 < x_2$ ,  $0 < y_1 < y_2$ , the cross-product ratio  $r = r(x_1, x_2; y_1, y_2) \geq 1$ . Without loss of generality, we consider only three possible cases below,

- (a)  $0 < x_1 \leq x_2 \leq y_1 \leq y_2$ , (b)  $0 < x_1 \leq y_1 \leq x_2 \leq y_2$ , (c)  $0 < x_1 \leq y_1 \leq y_2 \leq x_2$ ,

because the remaining cases can be proved by exchanging the roles of  $F$  and  $G$ .

For case (a), we have  $r \geq 1$  by the equivalence relations shown in the necessity part and by the continuity of  $H$  when  $x_2 = y_1$ . For case (b), the cross-product ratio

$$r = \frac{\exp(-\theta x_2) \overline{G}(y_1 - x_1) \overline{G}(y_2 - x_2)}{\exp(-\theta y_1) \overline{G}(y_2 - x_1) \overline{F}(x_2 - y_1)} \geq \frac{\overline{G}(y_1 - x_1) \overline{G}(y_2 - x_2) \overline{G}(x_2 - y_1)}{\overline{G}(y_2 - x_1)},$$

because  $\overline{F}(x_2 - y_1) \overline{G}(x_2 - y_1) \leq \exp(-\theta(x_2 - y_1))$  by the assumption. Recall that any IFR distribution is new better than used (Barlow and Proschan 1981, p.159). Therefore,  $\overline{G}(x + y) \leq \overline{G}(x) \overline{G}(y)$  for all  $x, y \geq 0$ , and hence the last  $r \geq 1$ . Similarly, for case (c),

$$r = \frac{\exp(-\theta y_2) \overline{G}(y_1 - x_1) \overline{F}(x_2 - y_2)}{\exp(-\theta y_1) \overline{G}(y_2 - x_1) \overline{F}(x_2 - y_1)} \geq \frac{\overline{G}(y_1 - x_1) \overline{G}(y_2 - y_1)}{\overline{G}(y_2 - x_1)} \times \frac{\overline{F}(y_2 - y_1) \overline{F}(x_2 - y_2)}{\overline{F}(x_2 - y_1)} \geq 1,$$

by the assumptions. This completes the proof.

Recall also that for any bivariate distribution  $H$  with marginals  $F$  and  $G$ , there exist a copula  $C$  (a bivariate distribution with uniform marginals on  $[0, 1]$ ) and a survival copula  $\hat{C}$  such that  $H(x, y) = C(F(x), G(y))$  and  $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))$  for all  $x, y \in R \equiv (-\infty, \infty)$ . Namely,  $C$  links  $H$  and  $(F, G)$ , while  $\hat{C}$  links  $\overline{H}$  and  $(\overline{F}, \overline{G})$ .

**Corollary 3.** Let  $(X, Y) \sim H = BLM(F, G, \theta)$ . Then the survival copula  $\hat{C}$  of  $H$  is  $TP_2$  iff the marginal distributions  $F$  and  $G$  are IFR and satisfy  $\overline{F}(x) \overline{G}(x) \leq \exp(-\theta x)$ ,  $x \geq 0$ .

**Proof.** Since the marginal  $F$  is absolutely continuous on the support  $[a_F, \infty)$  with positive density  $f$  on  $(a_F, \infty)$  (see Corollary 2(iii) above),  $F$  is strictly increasing and continuous on  $(a_F, \infty)$ . Similarly, the marginal  $G$  is strictly increasing and continuous on  $(a_G, \infty)$ . By Theorem 6, it suffices to prove that  $\overline{H}$  is  $TP_2$  on  $(a_F, \infty) \times (a_G, \infty)$  iff its survival copula  $\hat{C}$  is  $TP_2$  on  $(0, 1)^2$ . Recall that  $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))$ ,  $(x, y) \in (a_F, \infty) \times (a_G, \infty)$ , that  $\hat{C}(u, v) = \overline{H}(\overline{F}^{-1}(u), \overline{G}^{-1}(v))$ ,  $u, v \in (0, 1)$ , where  $\overline{F}^{-1}, \overline{G}^{-1}$  are inverse functions of  $\overline{F}, \overline{G}$ , respectively, and that all the functions  $\overline{F}, \overline{G}, \overline{F}^{-1}$  and  $\overline{G}^{-1}$  are decreasing. The required result then follows immediately (see, e.g., Lemma 5(ii) below).

The counterpart of  $TP_2$  property is the reverse regular of order two ( $RR_2$ ). For a real-valued function  $K$  on  $(a, b) \times (c, d)$ , we say that  $K$  is  $RR_2$  if the determinant of the  $2 \times 2$  matrix  $(K(x_i, y_j))$  is non-positive for all  $a < x_1 < x_2 < b$  and  $c < y_1 < y_2 < d$ . Mimicking the proof of Theorem 6, we conclude that for  $H = BLM(F, G, \theta)$ , the survival function  $\overline{H}$  is  $RR_2$  iff the survival copula  $\hat{C}$  of  $H$  is  $RR_2$  iff the marginal distributions  $F$  and  $G$  are DFR and satisfy  $\overline{F}(x) \overline{G}(x) \geq \exp(-\theta x)$ ,  $x \geq 0$ .

It is seen that all the conditions in Theorem 6 are satisfied by the Marshall–Olkin BVE. Therefore, the survival function and survival copula of the Marshall–Olkin BVE are both  $TP_2$ , regardless of parameters; a more general result will be given in Theorem 8 below. We next characterize, by a different approach, the  $TP_2$  property of some joint densities of absolutely continuous BLM distributions.

**Theorem 7.** Let  $H = BLM(F, G, \theta)$  be absolutely continuous and have joint density function  $h$ . Suppose that the marginal density functions  $f$  and  $g$  are three times differentiable on  $(0, \infty)$  and that  $\theta f(0^+) + f'(0^+) = \theta g(0^+) + g'(0^+)$  is finite. Assume further the functions

$$h_1(x|\theta) \equiv \theta f(x) + f'(x) > 0, \quad x > 0, \quad \text{and} \quad h_2(y|\theta) \equiv \theta g(y) + g'(y) > 0, \quad y > 0.$$

Then the joint density function  $h$  is  $TP_2$  iff (i)  $(h'_i(x|\theta))^2 \geq h''_i(x|\theta)h_i(x|\theta)$ ,  $x > 0$ ,  $i = 1, 2$ , and (ii)  $h_1(x|\theta)h_2(x|\theta) \leq h_1^2(0^+|\theta)\exp(-\theta x)$ ,  $x > 0$ .

To prove this theorem, we need the concept of local dependence function and the following lemma, which are essentially due to Holland and Wang (1987, p.872). An alternative (complete) proof of the lemma is provided below. In their proof, Holland and Wang (1987) assumed implicitly the *integrability* of the local dependence function, while Kemperman (1977, p.329) gave without proof the same result under *continuity* (smoothness) condition (see also Newman 1984). Wang (1993) proved that a positive continuous bivariate density on a Cartesian product  $(a, b) \times (c, d)$  is uniquely determined by its marginal densities and local dependence function when the latter exists and is integrable. On the other hand, the bivariate distributions with constant local dependence were investigated by Jones (1996, 1998).

**Lemma 4.** Let  $K$  be a positive function on  $D = (a, b) \times (c, d)$ . Then  $K$  is  $TP_2$  on  $D$  if and only if the local dependence function

$$\gamma_K(x, y) \equiv \frac{\partial^2}{\partial x \partial y} \log K(x, y) \geq 0 \quad \text{on } D,$$

provided the second-order partial derivatives exist.

**Proof.** Note that the following statements are equivalent: (a)  $\frac{\partial^2}{\partial x \partial y} \log K(x, y) \geq 0$  on  $D$ , (b)  $\frac{\partial}{\partial y} \log[K(x_2, y)/K(x_1, y)] \geq 0$  for all  $y$  and for all  $x_1 < x_2$ , (c)  $\log[K(x_2, y)/K(x_1, y)]$  is increasing in  $y$  for all  $x_1 < x_2$ , (d)  $K(x_2, y)/K(x_1, y)$  is increasing in  $y$  for all  $x_1 < x_2$ , (e)  $K(x_2, y_2)/K(x_1, y_2) \geq K(x_2, y_1)/K(x_1, y_1)$  for all  $y_1 < y_2$ ,  $x_1 < x_2$ , (f) the cross-product



ratio of  $K$  satisfies:  $K(x_1, y_1)K(x_2, y_2)/[K(x_1, y_2)K(x_2, y_1)] \geq 1$  for all  $x_1 < x_2$ ,  $y_1 < y_2$ , and (g) the function  $K$  is  $\text{TP}_2$  on  $D$ . The proof is complete.

**Proof of Theorem 7.** By the assumptions, the joint density function of  $H$  is of the form

$$h(x, y) = \begin{cases} e^{-\theta y} h_1(x - y|\theta), & x \geq y \\ e^{-\theta x} h_2(y - x|\theta), & x \leq y, \end{cases}$$

where  $h_i(0|\theta) \equiv h_i(0^+|\theta)$ ,  $i = 1, 2$ . For  $x \neq y$ , the local dependence function of  $h$  is

$$\gamma_h(x, y) = \frac{\partial^2}{\partial x \partial y} \log h(x, y) = \begin{cases} \frac{[h'_1(x-y|\theta)]^2 - h''_1(x-y|\theta)h_1(x-y|\theta)}{h_1^2(x-y|\theta)}, & x > y \\ \frac{[h'_2(y-x|\theta)]^2 - h''_2(y-x|\theta)h_2(y-x|\theta)}{h_2^2(y-x|\theta)}, & x < y. \end{cases}$$

Therefore,  $\gamma_h(x, y) \geq 0$  for all  $(x, y)$  with  $x \neq y$  iff the property (i) holds true.

(Necessity) If  $h$  is  $\text{TP}_2$  on  $(0, \infty)^2$ , then it is also  $\text{TP}_2$  on each rectangle (rectangular area) in the region  $\mathcal{A}_1 = \{(x, y) : x > y > 0\}$  or in  $\mathcal{A}_2 = \{(x, y) : y > x > 0\}$ , and hence the property (i) holds true by Lemma 4 and the above observation. Next, the property (ii) follows from the fact that for all  $0 < x_1 = y_1 < x_2 = y_2$ , the cross-product ratio  $r_h$  of  $h$  satisfies

$$1 \leq r_h \equiv r_h(x_1, x_2; y_1, y_2) = \frac{h(x_1, y_1)h(x_2, y_2)}{h(x_1, y_2)h(x_2, y_1)} = \frac{\exp(-\theta(x_2 - x_1))h_1(0|\theta)h_2(0|\theta)}{h_1(x_2 - x_1|\theta)h_2(x_2 - x_1|\theta)}.$$

This completes the proof of the necessity part.

(Sufficiency) Suppose  $0 < x_1 < x_2$  and  $0 < y_1 < y_2$ , then we want to prove the cross-product ratio  $r_h \geq 1$  under the assumptions (i) and (ii). If the rectangle with four vertices  $P_i, i = 1, 2, 3, 4$ , where  $P_1 = (x_1, y_1), P_2 = (x_2, y_1), P_3 = (x_2, y_2), P_4 = (x_1, y_2)$ , lies entirely in the region  $\mathcal{A}_1$  or  $\mathcal{A}_2$ , then  $r_h \geq 1$  by the assumption (i) and Lemma 4. If  $0 < x_1 = y_1 < x_2 = y_2$ , then the assumption (ii) implies  $r_h \geq 1$ . For the remaining cases, we apply the technique of factorization of the cross-product ratio if necessary. For example, if  $P_* = (x_1, y_*) \in \overline{P_1 P_4}$  and  $P^* = (x^*, y_2) \in \overline{P_4 P_3}$  denote the intersection of the diagonal line  $x = y$  and boundary of the rectangle, where  $x_1 < x^* < x_2$  and  $y_1 < y_* < y_2$ , then we split the original rectangle into four sub-rectangles by adding the new point  $(x^*, y_*)$  and calculate the ratio

$$r_h(x_1, x_2; y_1, y_2) = r_h(x_1, x^*; y_1, y_*)r_h(x_1, x^*; y_*, y_2)r_h(x^*, x_2; y_1, y_*)r_h(x^*, x_2; y_*, y_2) \geq 1,$$

each factor being greater than or equal to one by the previous results. The proof is complete.

It is known that for Marshall–Olkin BVE (7), both  $H$  and its joint survival function  $\overline{H}$  are PQD, so are its BVE copula  $C$ , survival function  $\overline{C}$  and survival copula  $\hat{C}$  (see Barlow

and Proschan 1981, p. 129). Moreover,  $\overline{H}$  and  $\hat{C}$  are  $\text{TP}_2$  due to Theorem 6 and its corollary (see also Nelsen 2006, p. 163, for a direct proof) and both are even  $\text{TP}_\infty$  if  $\lambda_1 = \lambda_2$  (Lin, Lai and Govindaraju 2016). We are now able to extend these results to the following.

**Theorem 8.** The Marshall–Olkin survival function  $\overline{H}$  and survival copula  $\hat{C}$  are both  $\text{TP}_\infty$ , regardless of parameters.

To prove this theorem, we need two more useful lemmas. Lemma 5 is well-known (see, e.g., Marshall, Olkin and Arnold 2011, p. 758), while Lemma 6 is essentially due to Gantmacher and Krein (2002), pp. 78–79 (see also Karlin 1968, p. 112, for an alternative version).

**Lemma 5.** Let  $r \geq 2$  be an integer.

- (i) If  $k(x, y)$  is  $\text{TP}_r$  in  $x$  and  $y$ , and if both  $u$  and  $v$  are nonnegative functions, then the product function  $K(x, y) = u(x)v(y)k(x, y)$  is  $\text{TP}_r$  in  $x$  and  $y$ .
- (ii) If  $k(x, y)$  is  $\text{TP}_r$  in  $x$  and  $y$ , and if  $u$  and  $v$  are both increasing, or both decreasing, then the composition function  $K(x, y) = k(u(x), v(y))$  is  $\text{TP}_r$  in  $x$  and  $y$ .

**Lemma 6.** Let  $\phi$  and  $\psi$  be two positive functions on  $(a, b)$ . Define the symmetric function  $K_s$  by

$$K_s(x, y) = \begin{cases} \psi(x)\phi(y), & a < y \leq x < b \\ \phi(x)\psi(y), & a < x \leq y < b. \end{cases}$$

If  $\phi(x)/\psi(x)$  is nondecreasing in  $x \in (a, b)$ , then the function  $K_s(x, y)$  is  $\text{TP}_\infty$  in  $x$  and  $y$ .

**Proof of Theorem 8.** Method (I). We prove first that the Marshall–Olkin survival function  $\overline{H}$  is  $\text{TP}_\infty$ . Rewrite the survival function (7) as

$$\begin{aligned} \overline{H}(x, y) &= \begin{cases} \exp[-(\lambda_1 + \lambda_{12})x - \lambda_2 y], & x \geq y \\ \exp[-(\lambda_2 + \lambda_{12})y - \lambda_1 x], & x \leq y \end{cases} \\ &= \exp[-\lambda_1 x - \lambda_2 y] K_s(x, y) \end{aligned}$$

where the symmetric function

$$K_s(x, y) = \begin{cases} \exp(-\lambda_{12}x), & x \geq y \\ \exp(-\lambda_{12}y), & x \leq y. \end{cases}$$

Let  $\phi(x) = 1$  and  $\psi(y) = \exp(-\lambda_{12}y)$ . Then by Lemma 6, we see that the function  $K_s$  is  $\text{TP}_\infty$ , so is  $\overline{H}$  by Lemma 5(i). Next, recall that the Marshall–Olkin survival copula

$$\hat{C}(u, v) = \overline{H}(\overline{F}^{-1}(u), \overline{G}^{-1}(v)), \quad u, v \in (0, 1),$$

where  $\overline{F}^{-1}$  and  $\overline{G}^{-1}$  are the inverse (decreasing) functions of  $\overline{F}(x) = \exp[-(\lambda_1 + \lambda_{12})x]$  and  $\overline{G}(y) = \exp[-(\lambda_2 + \lambda_{12})y]$ , respectively. Therefore,  $\hat{C}$  is  $\text{TP}_\infty$  by Lemma 5(ii).

Method (II). We prove first that the Marshall–Olkin survival copula  $\hat{C}$  is  $\text{TP}_\infty$ . Recall that the survival copula  $\hat{C}$  is

$$\hat{C}(u, v) = \begin{cases} u^{1-\phi}v, & u^\phi \geq v^\psi \\ uv^{1-\psi}, & u^\phi \leq v^\psi \end{cases} \quad (11)$$

(see, e.g., Lin, Lai and Govindaraju 2016), where  $\phi = \frac{\lambda_{12}}{\lambda_1 + \lambda_{12}} \in (0, 1)$  and  $\psi = \frac{\lambda_{12}}{\lambda_2 + \lambda_{12}} \in (0, 1)$ .

Let  $u_1 = u^\phi$  and  $v_1 = v^\psi$  in (11). Then it remains to prove that the new copula

$$\hat{C}_1(u_1, v_1) = \begin{cases} u_1^{1/\phi-1}v_1^{1/\psi}, & u_1 \geq v_1 \\ u_1^{1/\phi}v_1^{1/\psi-1}, & u_1 \leq v_1 \end{cases}$$

is  $\text{TP}_\infty$ . Further, rewrite  $\hat{C}_1(u_1, v_1) = u_1^{1/\phi}v_1^{1/\psi}C_2(u_1, v_1)$ , where the bivariate function

$$C_2(u_1, v_1) = \begin{cases} u_1^{-1}, & u_1 \geq v_1 \\ v_1^{-1}, & u_1 \leq v_1 \end{cases}$$

is clearly  $\text{TP}_\infty$  by Lemma 6. This together with Lemma 5 proves that both  $\hat{C}_1$  and  $\hat{C}$  are  $\text{TP}_\infty$ . Finally, recall that  $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))$ ,  $x, y \geq 0$ , where both  $\overline{F}$  and  $\overline{G}$  are decreasing functions. Therefore,  $\overline{H}$  is  $\text{TP}_\infty$  by Lemma 5(ii). This completes the proof.

It is well known that if a bivariate distribution  $H$  has  $\text{TP}_2$  density, then both  $H$  and its joint survival function  $\overline{H}$  are  $\text{TP}_2$  (see, e.g., Balakrishnan and Lai 2009, p.116). A more general result is given as follows.

**Theorem 9.** If the bivariate distribution  $H$  has  $\text{TP}_r$  density with  $r \geq 2$ , then both  $H$  and  $\overline{H}$  are  $\text{TP}_r$ . Consequently, if  $H$  has  $\text{TP}_\infty$  density, then both  $H$  and  $\overline{H}$  are  $\text{TP}_\infty$ .

**Proof.** Let us consider first the  $\text{TP}_\infty$  indicator functions  $K_1(x, y) = \mathbf{I}_{(-\infty, x]}(y)$  and  $K_2(x, y) = \mathbf{I}_{[x, \infty)}(y)$ , and then apply Theorem 3.5 of Gross and Richards (1998) restated below. For example, to prove the  $\text{TP}_r$  property of  $H$ , we have to claim that for all  $x_1 < \dots < x_r$  and  $y_1 < \dots < y_r$ , the determinant of each  $s \times s$  sub-matrix  $(H(x_i, y_j))$  (with  $2 \leq s \leq r$ ) is nonnegative. To prove this, let us recall that  $H(x_i, y_j) = E[\mathbf{I}_{(-\infty, x_i]}(X)\mathbf{I}_{(-\infty, y_j]}(Y)] = E[\phi(i, X)\psi(j, Y)]$ , where  $\phi(i, x) = \mathbf{I}_{(-\infty, x_i]}(x)$  is  $\text{TP}_r$  in two variables  $i \in \{1, 2, \dots, r\}$  and  $x \in R$ , and  $\psi(j, y) = \mathbf{I}_{(-\infty, y_j]}(y)$  is  $\text{TP}_r$  in two variables  $j \in \{1, 2, \dots, r\}$  and  $y \in R$ . Then Gross and Richards' Theorem applies and hence  $H$  is  $\text{TP}_r$ . Similarly, the survival function  $\overline{H}$  is  $\text{TP}_r$ . The proof is complete.

**Gross and Richards’ (1998) Theorem.** Let  $r \geq 2$  be an integer and let the bivariate  $(X, Y) \sim H$  have  $TP_r$  density. Assume further that both the functions  $\phi(i, x)$  and  $\psi(i, x)$  are  $TP_r$  in two variables  $i \in \{1, 2, \dots, r\}$  and  $x \in R$ . Then the  $r \times r$  matrix  $(E[\phi(i, X)\psi(j, Y)])$  is totally positive, that is, all its minors (of orders  $\leq r$ ) are nonnegative real numbers.

As mentioned in Balakrishnan and Lai (2009, p.124), the Block–Basu BVE (9) is PQD if  $\lambda_1 = \lambda_2$ . We now extend this result to the following.

**Theorem 10.** Suppose  $\lambda_1 = \lambda_2$  in the Block–Basu BVE. Then its joint density  $h$  is  $TP_\infty$ .

**Proof.** Take, in Lemma 6,  $\phi(x) = c_1 \exp(-\lambda_1 x)$  and  $\psi(y) = c_2 \exp[-(\lambda_2 + \lambda_{12})y]$  for some constants  $c_1, c_2 > 0$ . The proof is complete.

**Remark 3.** The same approach applies to other bivariate (non-BLM) distributions like Li and Pellerey’s (2011) generalized Marshall–Olkin bivariate distribution described below.

In Marshall and Olkin’s (1967) shock model:  $(X, Y) = (X_1 \wedge X_3, X_2 \wedge X_3)$ , we assume instead that  $X_1, X_2, X_3$  are independent general positive random variables (not limited to exponential ones) and that  $X_i \sim F_i$ ,  $i = 1, 2, 3$ . Let  $R_i = -\log \bar{F}_i$  be the hazard function of  $X_i$ . Then the generalized Marshall–Olkin bivariate distribution  $H$  has survival function of the form

$$\begin{aligned} \bar{H}(x, y) &= \Pr[X > x, Y > y] = \Pr[X_1 > x, X_2 > y, X_3 > \max\{x, y\}] \\ &= \exp[-R_1(x) - R_2(y) - R_3(\max\{x, y\})], \quad x, y \geq 0, \end{aligned} \quad (12)$$

which is PQD due to Li and Pellerey (2011). We now extend their result as follows.

**Theorem 11.** Let  $H$  be the generalized Marshall–Olkin distribution defined in (12). Then

- (i) the survival function  $\bar{H}$  is  $TP_\infty$ ;
- (ii) the survival copula  $\hat{C}$  of  $H$  is  $TP_\infty$ , provided the functions  $\bar{F}_1 \bar{F}_3$  and  $\bar{F}_2 \bar{F}_3$  are strictly decreasing.

**Proof.** Write the survival function

$$\begin{aligned} \bar{H}(x, y) &= \begin{cases} \exp[-(R_1(x) + R_3(x)) - R_2(y)], & x \geq y \\ \exp[-(R_2(y) + R_3(y)) - R_1(x)], & x \leq y \end{cases} \\ &= \exp[-R_1(x) - R_2(y)] K_s(x, y), \end{aligned}$$

where the symmetric function

$$K_s(x, y) = \begin{cases} \exp[-R_3(x)], & x \geq y \\ \exp[-R_3(y)], & x \leq y. \end{cases}$$

By taking  $\phi(x) = 1$  and  $\psi(y) = \exp[-R_3(y)]$  in Lemma 6, we know that the function  $K_s$  is  $\text{TP}_\infty$ , and hence the survival function  $\overline{H}$  is  $\text{TP}_\infty$  by Lemma 5(i). This proves part (i). To prove part (ii), we note that the marginal survival functions of  $H$  are  $\overline{F}(x) = \exp[-\tilde{R}_1(x)]$ ,  $x \geq 0$ , and  $\overline{G}(y) = \exp[-\tilde{R}_2(y)]$ ,  $y \geq 0$ , where the two functions  $\tilde{R}_1(x) = R_1(x) + R_3(x)$ ,  $x \geq 0$ , and  $\tilde{R}_2(y) = R_2(y) + R_3(y)$ ,  $y \geq 0$ , are strictly increasing by the assumptions on  $F_i$ ,  $i = 1, 2, 3$ . This in turn implies that the marginal survival functions  $\overline{F}$  and  $\overline{G}$  are strictly decreasing, so are their inverse functions. Therefore, the survival copula  $\hat{C}$  of  $H$ , which links  $\overline{H}$  and  $(\overline{F}, \overline{G})$ , is also  $\text{TP}_\infty$  by part (i) and Lemma 5(ii). This completes the proof.

## 6. Stochastic Comparisons of Bivariate LM Distributions

Denote by  $\mathcal{BLM}$  the family of all BLM distributions, namely,

$$\mathcal{BLM} = \{H : H = \text{BLM}(F, G, \theta), \text{ where } \theta > 0, \text{ and } F, G \text{ are marginal distributions}\}.$$

Then we can study stochastic comparisons in the  $\mathcal{BLM}$  family. For example, we have the following results whose proofs are straightforward and are omitted.

**Theorem 12.** Let  $(X_i, Y_i) \sim H_i = \text{BLM}(F_i, G_i, \theta_i)$ ,  $i = 1, 2$ . Then we have

- (i)  $X_1 \leq_{st} X_2$ ,  $Y_1 \leq_{st} Y_2$  and  $\theta_1 \geq \theta_2$  iff  $(X_1, Y_1) \leq_{uo} (X_2, Y_2)$  (in the upper orthant order), namely,  $\overline{H}_1(x, y) \leq \overline{H}_2(x, y) \forall x, y \geq 0$ , or, equivalently,  $E[K(X_1, Y_1)] \leq E[K(X_2, Y_2)]$  for any bivariate distribution  $K$  on  $R_+^2$ ;
- (ii)  $F_1 = F_2$ ,  $G_1 = G_2$  and  $\theta_1 \geq \theta_2$  iff  $(X_1, Y_1) \leq_c (X_2, Y_2)$  (in the concordance order), namely,  $E[k_1(X_1)k_2(Y_1)] \leq E[k_1(X_2)k_2(Y_2)]$  for all increasing functions  $k_1, k_2 \geq 0$ ; and
- (iii) if  $X_1 \leq_{Lt} X_2$ ,  $Y_1 \leq_{Lt} Y_2$  and  $\theta_1 = \theta_2$ , then  $(X_1, Y_1) \leq_{Lt} (X_2, Y_2)$  (in the Laplace transform order), namely,  $\mathcal{L}_1(s, t) \geq \mathcal{L}_2(s, t)$  for all  $s, t \geq 0$ , or, equivalently,  $E[k_1(X_1)k_2(Y_1)] \geq E[k_1(X_2)k_2(Y_2)]$  for all completely monotone functions  $k_1, k_2$ .

When  $H_1$  and  $H_2$  have the same pair of marginals  $(F, G)$ , Theorem 12(i) reduces, by Corollary 1, to the following interesting result which is related to the famous Slepian's inequality on the bivariate normal distributions (see the discussion in Remark 4 below).

**Corollary 4.** Let  $(X_i, Y_i) \sim H_i = BLM(F, G, \theta_i)$ ,  $i = 1, 2$ , with correlations  $\rho_1$  and  $\rho_2$ , respectively. Then  $\rho_1 \leq \rho_2$  iff  $\overline{H}_1(x, y) \leq \overline{H}_2(x, y) \forall x, y \geq 0$ , or, equivalently,  $H_1(x, y) \leq H_2(x, y) \forall x, y \geq 0$ .

**Remark 4.** In Corollary 4 above, if we consider standard bivariate normal distributions instead of BLM ones, then the conclusion also holds true and the necessary part is the so-called Slepian's lemma/inequality; see Slepian (1962), Müller and Stoyan (2002), p.97, and Hoffmann-Jørgensen (2013) for more general results. In Wikipedia, it was said that while this intuitive-seeming result is true for Gaussian processes, it is not in general true for other random variables. However, as we can see in Corollary 4, there are infinitely many BLM distributions sharing the same Slepian's inequality with bivariate normal ones.

**Remark 5.** Consider a two-component system and let the two components have lifetimes  $(X, Y) \sim H = BLM(F, G, \theta)$ . Then the lifetime of a series system composed of these two components is  $X \wedge Y \sim Exp(\theta)$ , while the lifetime of a parallel system composed of these two components is  $X \vee Y$  obeying the distribution  $H_p(z) = e^{-\theta z} - 1 + F(z) + G(z)$ ,  $z \geq 0$ . Therefore the mean times to failure of series and parallel systems are, respectively,  $E[X \wedge Y] = \int_0^\infty \overline{H}(x, x)dx = 1/\theta$  and  $E[X \vee Y] = E(X) + E(Y) - \int_0^\infty \overline{H}(x, x)dx = E(X) + E(Y) - 1/\theta$  (see Lai and Lin 2014 for details and more general results).

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